Existence of Anosov diffeomorphisms on infra-nilmanifolds modeled on free nilpotent Lie groups

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Abstract

An infra-nilmanifold is a manifold which is constructed as a quotient space $\Gamma \backslash G$ of a simply connected nilpotent Lie group G, where Γ is a discrete group acting properly discontinuously and cocompactly on G via so called affine maps. The manifold $\Gamma \backslash G$ is said to be modeled on the Lie group G. This class of manifolds is conjectured to be the only class of closed manifolds allowing an Anosov diffeomorphism. However, it is far from obvious which of these infra–nilmanifolds actually do admit an Anosov diffeomorphism. In this paper we completely solve this question for infra-nilmanifolds modeled on a free c–step nilpotent Lie group.

1 Anosov diffeomorphisms on infra–nilmanifolds

There has been quite some activity in the study of Anosov diffeomorphisms on closed manifolds in the last ten to fifteen years ([1, 2, 3, 4, 5, 6, 7, 8, 11, 12, 13, 14, 15, 16]). Vaguely said, an Anosov diffeomorphism of a manifold M is a diffeomorphism $f : M \to M$ such that the tangent bundle of M allows a df–invariant continuous splitting $TM = E^s \oplus E^u$ with df contracting on E^s and expanding on E^u. Up till now, the only examples of closed manifolds admitting an Anosov diffeomorphism are manifolds which are homeomorphic to an infra–nilmanifold. In fact, it has been conjectured that this is the only class of closed manifolds admitting such diffeomorphisms. We refer to the papers cited above for more information.

Let us briefly recall what an infra–nilmanifold is and how an Anosov diffeomorphism can be constructed on such a manifold. Let G be a connected and simply connected nilpotent Lie group and $\mathrm{Aut}(G)$ the group of continuous automorphisms of G. The affine group $\mathrm{Aff}(G)$ is defined as the semi-direct product $G \rtimes \mathrm{Aut}(G)$ and acts on G in the following way:

$$\forall \alpha = (g, \delta) \in \mathrm{Aff}(G),\ \forall h \in G : \quad {}^{\alpha}h = g\delta(h).$$

Let $C \subseteq \mathrm{Aut}(G)$ be a compact subgroup of automorphisms. A subgroup $\Gamma \subseteq G \rtimes C$ is called an almost Bieberbachgroup if Γ is a discrete, torsion-free subgroup such that the quotient $\Gamma \backslash G$ is compact. The quotient space $\Gamma \backslash G$ is a closed manifold and is called an infra-nilmanifold modeled on the Lie group G. Let $p : \Gamma \to \mathrm{Aut}(G)$ denote the natural projection on the second component, then it is well known that $H = p(\Gamma)$ is a finite group, which is called the holonomy group of Γ. The subgroup $N = \Gamma \cap G$ is a uniform lattice in G and Γ fits in the following exact sequence:

$$1 \longrightarrow N \longrightarrow \Gamma \longrightarrow H \longrightarrow 1 \ . \tag{1}$$

In the case where G is abelian, i.e. $G \cong \mathbb{R}^n$ for some n, the manifolds constructed in this way are exactly the compact flat Riemannian manifolds.

Let $\alpha \in \mathrm{Aff}(G)$ be an affine transformation such that $\alpha\Gamma\alpha^{-1} = \Gamma$. Then α induces a diffeomorphism $\bar{\alpha}$ on the infra–nilmanifold $\Gamma \backslash G$, which is defined by

$$\bar{\alpha} : \Gamma\backslash G \to \Gamma\backslash G : \Gamma g \mapsto \Gamma(^{\alpha}g).$$

A diffeomorphism like this is called an affine infra-nilmanifold automorphism. The map $\bar{\alpha}$ is an Anosov diffeomorphism if and only if the linear part of α is hyperbolic, i.e. it only has eigenvalues of absolute value different from 1. It is conjectured that every Anosov diffeomorphism is topologically conjugate to an affine infra-nilmanifold automorphism (see also [3]).

In [17], H. L. Porteous gives an easy criterion to decide whether or not a flat manifold allows an Anosov diffeomorphism. To formulate this criterion, we have to introduce the holonomy representation. For a flat manifold, the short exact sequence (1) is of the form $1 \to \mathbb{Z}^n \to \Gamma \to H \to 1$ and hence conjugation in Γ induces a representation $\varphi : H \to \mathrm{Aut}(\mathbb{Z}^n) = \mathrm{GL}_n(\mathbb{Z})$. It is this representation which is called the holonomy representation. The criterion of Porteous states that a flat manifold M supports an Anosov diffeomorphisms if and only if every $\mathbb{Q}$-irreducible component of φ which occurs with multiplicity 1, splits over $\mathbb{R}$. (We can also view φ as a rational representation $\varphi : H \to \mathrm{GL}_n(\mathbb{Q})$ and a real representation $\varphi : H \to \mathrm{GL}_n(\mathbb{R})$.) As it was already pointed out in [5], a natural class to consider for possibly generalizing this statement is the class of infra–nilmanifolds modeled on a free c–step nilpotent Lie group, i.e. Lie groups where the corresponding Lie algebra is free c–step nilpotent. The flat case is then the case where $c = 1$.

For infra–nilmanifolds which are not flat (so which are not modeled on an abelian Lie group), the short exact sequence (1) does not give rise to a natural holonomy representation $\varphi : H \to \mathrm{Aut}(N)$. Therefore, a rational holonomy representation was introduced in [5]. To obtain this rational holonomy representation, one embeds the group N into its rational Mal'cev completion $N_{\mathbb{Q}}$ (or radicable hull). This leads to the following commutative diagram:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & N & \longrightarrow & \Gamma & \longrightarrow & H & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & N_{\mathbb{Q}} & \longrightarrow & \Gamma_{\mathbb{Q}} & \longrightarrow & H & \longrightarrow & 1,
\end{array}
$$

where the bottom exact sequence splits. By fixing a splitting morphism $s : H \to \Gamma_{\mathbb{Q}}$, we define the rational holonomy representation $\varphi : H \to \mathrm{Aut}(N_{\mathbb{Q}})$ by

$$\varphi(f)(n) = s(f)ns(f)^{-1}.$$

The rational holonomy representation does depend on the choice of s, but this dependence is not relevant in the study of Anosov diffeomorphisms. In fact, it turns out that the rational holonomy representation contains all information about the existence of Anosov diffeomorphisms, see e.g. [5, Theorem A]. The representation φ also induces a representation

$$\bar{\varphi} : H \to \mathrm{Aut}(N_{\mathbb{Q}}/[N_{\mathbb{Q}}, N_{\mathbb{Q}}]) \cong \mathrm{GL}_n(\mathbb{Q}) \quad (\text{for some } n),$$

which will be refered to as the abelianized rational holonomy representation. Our main theorem, which generalizes the criterion of Porteous and completely finishes the work started in [5], states that we can decide whether or not an infra-nilmanifold modeled on a free c–step nilpotent Lie group has an Anosov diffeomorphism by only looking at the abelianized rational holonomy representation $\bar{\varphi}$:

Theorem 1.1. *Let M be an infra-nilmanifold modeled on a free c-step nilpotent Lie group, with holonomy group H and associated abelianized rational holonomy representation $\overline{\varphi} : H \to \mathrm{Aut}\left(\frac{N_{\mathbb{Q}}}{[N_{\mathbb{Q}}, N_{\mathbb{Q}}]}\right)$. Then the following are equivalent:*

$$M \text{ admits an Anosov diffeomorphism.}$$
$$\Updownarrow$$
Every $\mathbb{Q}$-irreducible component of $\bar{\varphi}$ that occurs with multiplicity m, splits in more than $\frac{c}{m}$ components when seen as a representation over $\mathbb{R}$.

For abelian holonomy groups H, this was already shown in [7, Theorem 2.3]. From [7], we know that the existence of an Anosov diffeomorphisms is equivalent with the existence of a c-hyperbolic, integer-like matrix which commutes with every element in the image of $\overline{\varphi}$. Recall that a matrix $C \in \mathrm{GL}_n(\mathbb{Q})$ is called integer-like if its characteristic polynomial has coefficients in $\mathbb{Z}$ and its determinant has absolute value equal to 1. A matrix is called c-hyperbolic if there does not exists a natural number $k \leq c$ such that the product of any k (not necessarily different) eigenvalues of the matrix has absolute value equal to 1. By decomposing the representation $\overline{\varphi}$ into its $\mathbb{Q}$-irreducible components, it was shown in [7] that the following theorem implies our main theorem:

Theorem 1.2. *Let $\rho : G \to \mathrm{GL}_n(\mathbb{Q})$ be a $\mathbb{Q}$-irreducible representation. Then there exists a c-hyperbolic, integer-like matrix $C \in \mathrm{GL}_{mn}(\mathbb{Q})$ which commutes with $m\rho = \underbrace{\rho \oplus \rho \oplus \cdots \oplus \rho}_{m \text{ times}}$ if and only if ρ splits in strictly more than $\frac{c}{m}$ components when seen as a representation over $\mathbb{R}$.*

In the second section we show the first direction of our main theorem by just looking at the dimension of the real components of a $\mathbb{Q}$-irreducible representation. The other direction is harder and we will need some preliminary results about number theory and splitting fields of representations. The work for this direction is done in the third section. In the last section we shortly explain how the results of this paper can be generalized to some more classes of infranilmanifolds. Throughout this paper, we will use results from representation theory of finite groups. We will give explicit references at several places. When needed, the reader can consult [10] (mainly chapters 9 and 10) and [18] for more information.

2   A condition on the number of components

All fields we will consider in this paper are assumed to be subfields of the field $\mathbb{C}$ of complex numbers. Let G be a finite group and $\mathrm{Irr}(G)$ the set of $\mathbb{C}$-irreducible characters of G. Let F be a field and χ a character (not necessarily irreducible), then we say that χ is afforded by a F-representation if there exists a representation $\rho : G \to \mathrm{GL}_n(F)$ such that the character of ρ equals χ. For each $\chi \in \mathrm{Irr}(G)$, we can define the complex number

$$\nu_2(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2).$$

It is well known (see e.g. [10, page 58] or [18, Prop. 39, page 109]) that $\nu_2(\chi)$ can only have values $1, 0$ and -1 with the following possibilities:

- $\nu_2(\chi) = 0$ if and only if χ is not real (i.e. $\exists g \in G$ such that $\chi(g) \notin \mathbb{R}$).

- $\nu_2(\chi) = 1$ if and only if χ is afforded by a real representation.

- $\nu_2(\chi) = -1$ if and only if χ is real but not afforded by a real representation.

If χ is an irreducible character, we write $\mathbb{Q}(\chi)$ for the smallest subfield of $\mathbb{C}$ that contains $\chi(g)$ for all $g \in G$. The field $K = \mathbb{Q}(\chi)$ is Galois over $\mathbb{Q}$ and for every $\sigma \in \mathrm{Gal}(K, \mathbb{Q})$, we can define the map

$$\chi^\sigma : G \to K : g \mapsto \sigma(\chi(g)),$$

which we call a Galois conjugate of χ. Every Galois conjugate of χ is also an irreducible character of G (see [10, Lemma 9.16]). It is easy to show that the invariant ν_2 is the same for all of these Galois conjugates:

Lemma 2.1. *Let $\chi \in \mathrm{Irr}(G)$ and $\sigma \in \mathrm{Gal}(\mathbb{Q}(\chi), \mathbb{Q})$, then we have that $\nu_2(\chi) = \nu_2(\chi^\sigma)$.*

Proof. By using the formula for ν_2, we have

$$
\begin{aligned}
\nu_2(\chi^\sigma) = \frac{1}{|G|} \sum_{g \in G} \chi^\sigma(g^2) &= \frac{1}{|G|} \sum_{g \in G} \sigma(\chi(g^2)) \\
&= \sigma(\frac{1}{|G|} \sum_{g \in G} \chi(g^2)) \\
&= \sigma(\nu_2(\chi)) = \nu_2(\chi)
\end{aligned}
$$

since $\nu_2(\chi) \in \mathbb{Z}$. $\qquad\square$

This easy lemma has the following important corollary about the dimension of irreducible components:

Corollary 2.2. *All $\mathbb{R}$-irreducible components of a $\mathbb{Q}$-irreducible representation have the same dimension.*

Proof. Let χ be the character of an irreducible component of the $\mathbb{Q}$-irreducible representation ρ.

If $\nu_2(\chi) = 1$, then all irreducible components of ρ are afforded by a real representation, since they are all of the form χ^σ for some $\sigma \in \mathrm{Gal}(\mathbb{Q}(\chi), \mathbb{Q})$ ([10, Lemma 9.21 (c)]). Thus the $\mathbb{R}$-irreducible components of ρ are equal to the $\mathbb{C}$-irreducible components and they all have the same dimension $\chi(e_G)$.

If $\nu_2(\chi) \neq 1$, then none of the irreducible components is afforded by a real representation and thus every $\mathbb{R}$-irreducible component has the same dimension $\chi(e_G) + \overline{\chi}(e_G) = 2\chi(e_G)$. $\qquad\square$

The next proposition follows naturally from the corollary:

Proposition 2.3. *Let $\rho : G \to GL_{kr}(\mathbb{Q})$ be a $\mathbb{Q}$-irreducible representation with r the dimension of every $\mathbb{R}$-irreducible component of ρ. If $C \in GL_{krm}(\mathbb{Q})$ commutes with $m\rho = \rho \oplus \rho \oplus \ldots \oplus \rho$ and $|\det(C)| = 1$, then C cannot be km-hyperbolic.*

Proof. By using the real Jordan canonical form of C (see e.g. [9, Theorem 3.4.5.]), we find an invertible matrix $A \in GL_{krm}(\mathbb{R})$ such that

$$
ACA^{-1} = \begin{pmatrix} C_1 & 0 & \ldots & 0 \\ 0 & C_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & C_l \end{pmatrix}
$$

where C_i and C_j have distinct eigenvalues for $i \neq j$ and such that C_i either has one real eigenvalue or two complex conjugate eigenvalues. Let λ_i be one (of the at most two) eigenvalue(s) of C_i. So all λ_i are distinct by construction. By conjugating with the same matrix A, the representation $m\rho$ also splits over $\mathbb{R}$, since the generalized eigenspaces of C are obviously invariant under the representation $m\rho$. This implies that each C_i has dimension $k_i r$ for some $k_i \in \mathbb{N}_0$, because all real components of $m\rho$ have the same dimension r. So for the determinant of C, we have

$$
\begin{aligned}
1 = |\det(C)| = \prod_{i=1}^{l} |\det(C_i)| &= \prod_{i=1}^{l} |\lambda_i|^{k_i r} \\
&= \left(\prod_{i=1}^{l} |\lambda_i|^{k_i} \right)^r
\end{aligned}
$$

and therefore $\prod_{i=1}^{l} |\lambda_i|^{k_i} = 1$. This means that C is not km hyperbolic, since $\sum_{i=1}^{l} k_i = km$ by looking at the size of C. $\qquad\square$

This theorem implies the first direction of the main theorem. Indeed, if there exists a c-hyperbolic, integer-like matrix that commutes with $m\rho$, then the k of Proposition 2.3 must be strictly larger than $\frac{c}{m}$. Note that this k is the number of $\mathbb{R}$-irreducible components of ρ and thus gives us the first direction of Theorem 1.2.

4

3 Construction of an Anosov diffeomorphism

The proof of the other direction needs some more work and uses different results of number theory and representation theory for finite groups. The first step is investigating how a $\mathbb{Q}$-irreducible representation splits over certain fields, especially over minimal splitting fields. Next, we consider the existence of c-hyperbolic units in number fields, which correspond to c-hyperbolic matrices. By combining these results, we can construct a c-hyperbolic, integer-like matrix C which commutes with the given representation.

3.1 Decomposition over a minimal splitting field

If ρ is a $\mathbb{Q}$-irreducible representation of an abelian group, then its image is always cyclic. A consequence is that in this case every $\mathbb{Q}$-irreducible representation has a basis of eigenvectors which are Galois conjugates. In this section we answer the question how a general representation splits over a minimal splitting field. For the formulation of this result we will make use of permutation matrices.

Let $K = \mathbb{Q}(\theta) \supseteq \mathbb{Q}$ be an extension of degree n and $\{\sigma_1, \ldots, \sigma_n\}$ the set of monomorphisms $K \to \mathbb{C}$. We say that σ_i is real if $\sigma_i(K) \subseteq \mathbb{R}$, otherwise we call σ_i complex. If σ_i is complex then $\overline{\sigma_i}$ is a different monomorphism $K \to \mathbb{C}$, so the complex monomorphisms come in pairs. We conclude that $n = s + 2t$ with s the number of real monomorphisms and $2t$ the number of complex monomorphisms. Let us remark that every monomorphism σ_i is completely determined by the image of θ, namely $\theta_i = \sigma_i(\theta)$. If $t = 0$, so if every σ_i has a real image, then we call the field K totally real. When $s = 0$ we will say that the field is totally imaginary.

Let ρ be a $\mathbb{Q}$-irreducible representation. From [10, Theorem 9.21] we know that each $\mathbb{C}$-irreducible component of ρ occurs with the same multiplicity m. If $\chi \in \mathrm{Irr}(G)$ is the character of one of those components, then the set of characters of all $\mathbb{C}$-irreducible components of ρ is given by the Galois conjugates of χ. So if we have $\{\sigma_1, \ldots, \sigma_n\} = \mathrm{Gal}(\mathbb{Q}(\chi), \mathbb{Q})$, we can write the character χ_ρ of ρ as

$$\chi_\rho = m\chi^{\sigma_1} \oplus \ldots \oplus m\chi^{\sigma_n}.$$

The fixed multiplicity m is called the Schur Index of χ over $\mathbb{Q}$ and is written as $m_{\mathbb{Q}}(\chi)$ (see [10, pages 160–161]). The number $m_{\mathbb{Q}}(\chi)$ is also the smallest integer m such that $m\chi$ is afforded by a $\mathbb{Q}(\chi)$–representation. Moreover, it is also given by the minimal degree of a field extension $\mathbb{Q}(\chi) \subseteq F$ such that χ is afforded by an F-representation ([10, Theorem 10.17]). Such a field F is called a minimal splitting field for the character χ. Note that the number of $\mathbb{C}$-irreducible components of ρ is equal to $m \cdot n = [F : \mathbb{Q}(\chi)] \cdot [\mathbb{Q}(\chi) : \mathbb{Q}] = [F : \mathbb{Q}]$.

Let $\pi \in S_n$ be a permutation, then there exists a permutation matrix $K_\pi \in \mathrm{GL}_n(\mathbb{Z})$, which is defined by

$$(K_\pi)_{ij} = \begin{cases} 1 & j = \pi(i) \\ 0 & \text{otherwise} \end{cases}$$

Note that the relation $K_{\pi_1} K_{\pi_2} = K_{\pi_2 \pi_1}$ holds for all $\pi_1, \pi_2 \in S_n$, so in particular we have that $K_\pi^{-1} = K_{\pi^{-1}}$. Let M be a matrix in $GL_n(\mathbb{C})$, then $K_\pi M$ is the matrix M where the rows are permuted according to π, so the i-th row of $K_\pi M$ is the $\pi(i)$-th row of M. If we write

$$M = \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{pmatrix},$$

as a column of row vectors, then another way of saying this is that

$$K_\pi M = \begin{pmatrix} m_{\pi(1)} \\ m_{\pi(2)} \\ \vdots \\ m_{\pi(n)} \end{pmatrix}.$$

In the same way, MK_π is the matrix M where the columns are permuted according to π^{-1}. This last property can easily be checked by observing that the transpose $(K_\pi)^T = K_{\pi^{-1}}$ and then writing $MK_\pi = (K_{\pi^{-1}}M^T)^T$. The matrix $K_\pi \otimes \mathbb{1}_k \in \mathrm{GL}_{kn}(\mathbb{Z})$ will be denoted as $K_\pi^{\otimes k}$. It has the same interpretation as K_π, but by using blocks of k rows or columns instead.

Let $\mathbb{Q} \subseteq E$ be a field extension of finite degree such that E is Galois over $\mathbb{Q}$. Take a subfield $F = \mathbb{Q}(\theta) \subseteq E$, which is not necessarily Galois over $\mathbb{Q}$, and denote by $\sigma_1, \ldots, \sigma_n$ the n distinct monomorphisms $F \to \mathbb{C}$. Because $\theta_i = \sigma_i(\theta) \in E$, we have that $\sigma_i(F) \subseteq E$ for all i. It follows that for any $\sigma \in \mathrm{Gal}(E, \mathbb{Q})$ and any $i \in \{1, 2, \ldots, n\}$, we have that $\sigma \circ \sigma_i : F \to E \subseteq \mathbb{C}$ is again a monomorphism of F in $\mathbb{C}$. Hence σ induces a permutation on $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$. Associated to this is a permutation $\pi_\sigma \in S_n$ which is determined by

$$\sigma \circ \sigma_i = \sigma_{\pi_\sigma(i)}.$$

We let $K_\sigma \in \mathrm{GL}_n(\mathbb{Z})$ denote the corresponding permutation matrix.

We already mentioned above that for every $\mathbb{Q}$-irreducible representation ρ of an abelian group, one can always find a basis of eigenvectors which are Galois-conjugates, see [7]. This means that if E is a minimal splitting field of ρ with $\mathrm{Gal}(E, \mathbb{Q}) = \{\sigma_1, \ldots, \sigma_n\}$, there exists a vector $v_0 \in E^n$ such that with respect to the basis $\{v_1 = \sigma_1(v_0), v_2 = \sigma_2(v_0), \ldots, v_n = \sigma_n(v_0)\}$, the representation is diagonal.

The main part of this subsection is to generalize this statement to arbitrary groups:

Theorem 3.1. *Let $\chi \in \mathrm{Irr}(G)$ be the character of an irreducible component of a $\mathbb{Q}$-irreducible representation ρ. Let $\mathbb{Q} \subseteq F$ be a field extension of minimal degree such that χ is afforded by a F-representation, say ρ_0 and assume that k is the dimension of this representation. If $[F : \mathbb{Q}] = n$ and $\sigma_1, \ldots, \sigma_n$ are the n distinct monomorphisms $F \to \mathbb{C}$ and if E is any field extension $\mathbb{Q} \subseteq F \subseteq E$ such that E is Galois over $\mathbb{Q}$, then there exist a $P \in \mathrm{GL}_{kn}(E)$ such that the following conditions hold:*

(1) $P^{-1}\rho P = \begin{pmatrix} \sigma_1(\rho_0) & 0 & \ldots & 0 \\ 0 & \sigma_2(\rho_0) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \sigma_n(\rho_0) \end{pmatrix}.$

(2) For all $\sigma \in \mathrm{Gal}(E, \mathbb{Q})$: $\sigma(P) = PK_{\sigma^{-1}}^{\otimes k}.$

Proof. Let $\rho_0 : G \to GL_k(F)$ be an irreducible representation with character χ. Take the representations $\rho^i = \sigma_i(\rho_0) : G \to \mathrm{GL}_k(\sigma_i(F))$ with character χ^{σ_i}. We construct a new representation

$$\tilde{\rho} : G \to \mathrm{GL}_{kn}(E) : g \mapsto \begin{pmatrix} \rho_g^1 & 0 & \ldots & 0 \\ 0 & \rho_g^2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \rho_g^n \end{pmatrix},$$

which was already used in the formulation of the theorem.

Since F is a finite field extension of $\mathbb{Q}$, we know that $F = \mathbb{Q}(\theta)$ for some algebraic number θ. Look at the matrix

$$Q = \begin{pmatrix} \sigma_1(1) & \sigma_1(\theta) & \ldots & \sigma_1(\theta^{n-1}) \\ \sigma_2(1) & \sigma_2(\theta) & \ldots & \sigma_2(\theta^{n-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_n(1) & \sigma_n(\theta) & \ldots & \sigma_n(\theta^{n-1}) \end{pmatrix} \otimes \mathbb{1}_k,$$

then it follows by construction that $\sigma(Q) = K_\sigma^{\otimes k}Q$. Note that Q is the Kronecker product of a Vandermonde matrix with the identity and thus its determinant can easily be computed and is

different from 0. So Q is invertible and denote the inverse of Q by P. By applying $\sigma \in \mathrm{Gal}(E, \mathbb{Q})$ to the relation $QP = \mathbb{1}_{kn}$, we find that

$$\sigma(P) = \sigma(Q)^{-1} = Q^{-1}(K_\sigma^{\otimes k})^{-1} = PK_{\sigma^{-1}}^{\otimes k}$$

and thus the matrix P satisfies condition 2 of the theorem.

First we show that $P\tilde{\rho}_g Q \in \mathrm{GL}_{nk}(\mathbb{Q})$ for all $g \in G$. Equivalently, we have to show that $\sigma(P\tilde{\rho}_g Q) = P\tilde{\rho}_g Q$ for all $\sigma \in \mathrm{Gal}(E, \mathbb{Q})$. We compute that

$$\sigma(P\tilde{\rho}_g Q) = \sigma(P)\sigma(\tilde{\rho}_g)\sigma(Q) = PK_{\sigma^{-1}}^{\otimes k}\sigma(\tilde{\rho}_g)K_\sigma^{\otimes k}Q.$$

It is easy to see that $\sigma(\tilde{\rho}_g) = K_\sigma^{\otimes k}\tilde{\rho}_g K_{\sigma^{-1}}^{\otimes k}$ and thus the conclusion follows. So it suffices to show that the representations $P\tilde{\rho}Q$ and ρ are equivalent over $\mathbb{Q}$.

Note that $P\tilde{\rho}Q$ and ρ have a common irreducible factor when seen as representation over $\mathbb{C}$ (namely ρ_0, corresponding to the character χ). Also, it follows from the discussion above about the characters of $\mathbb{Q}$-irreducible representations that both have the same dimension, namely kn. As a consequence of [10, Corollary 9.7], we have that $P\tilde{\rho}Q$ has a $\mathbb{Q}$-irreducible component which is equivalent with ρ over $\mathbb{Q}$. Because they have the same dimension, this ends the proof of the theorem. $\square$

It's easy to see that this theorem is in fact a generalization of the statement for abelian groups. The advantage of working with a matrix P that satisfies the conditions of the theorem is that we can easily construct matrices that have coefficients in $\mathbb{Q}$:

Proposition 3.2. *Let $F \supseteq \mathbb{Q}$ be a field extension of degree n and $\sigma_1, \ldots, \sigma_n$ the distinct monomorphisms from F to $\mathbb{C}$. Let C_0 be any matrix with coefficients in F and look at the matrix*

$$C = \begin{pmatrix} \sigma_1(C_0) & 0 & \cdots & 0 \\ 0 & \sigma_2(C_0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n(C_0) \end{pmatrix}.$$

Let E be any field extension of F which is Galois over $\mathbb{Q}$ and P an invertible matrix with entries in E which satisfies

$$\sigma(P) = PK_{\sigma^{-1}}^{\otimes k}$$

for all $\sigma \in \mathrm{Gal}(E, \mathbb{Q})$, where we use notations as above. Then it holds that PCP^{-1} is a matrix with coefficients in $\mathbb{Q}$.

The proof of this proposition is immediate. Note that the matrix C of the previous proposition doesn't depend on the choice of Galois extension E. Also the construction of the matrix P in the proof of Theorem 3.1 didn't depend on the field E. It was only used to embed the monomorphisms σ_i in a nice group structure. In the rest of this article we will ignore the use of the Galois extension E.

3.2 Existence of real minimal splitting fields

For a $\mathbb{Q}$-irreducible representation ρ of a finite cyclic group, there is always a canonical choice of a minimal field extension $F \supseteq \mathbb{Q}$ such that the representation is completely reducible over F, i.e. is diagonalizable over F. If $f \in \mathbb{Q}[X]$ is the characteristic polynomial of a generator of the image of ρ, then the field F is equal to the splitting field of f over $\mathbb{Q}$. For non-abelian groups, we explained above that such a minimal splitting field F also exists, but it is far from unique in general. In this subsection we show how to construct a real minimal splitting field in the case where ρ is completely reducible over $\mathbb{R}$.

An example of the non-uniqueness of the minimal splitting field can be given by the unique two-dimensional irreducible representation of the quaternion group Q_8. Take complex numbers $\alpha, \beta \in \mathbb{C}$ with $\alpha^2 + \beta^2 = -1$ and look at the representation ρ given by

$$\rho(-1) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\rho(i) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\rho(j) = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}.$$

It's an easy exercise to check that this indeed defines a representation of Q_8. This shows that every field of the form $\mathbb{Q}(\sqrt{-1 - \alpha^2})$ with $\alpha \in \mathbb{Q}$ is a minimal splitting field for this representation. These fields do have in common that they are not real. In fact it's an exercise to show that every minimal splitting field for this character is totally imaginary.

In general it's not true that if a character χ is afforded by an E-representation for a field $E \supseteq \mathbb{Q}$, that there is also a subfield $F \subseteq E$ such that F has minimal degree and χ is afforded by a F-representation. Therefore we cannot directly conclude that every real representation has a real minimal splitting field. If $m_{\mathbb{Q}}(\chi) = 1$, or said differently, if $\mathbb{Q}(\chi)$ is a splitting field for ρ, then this must of course be true. By the Brauer-Speiser Theorem, see for example [10, page 171], we know that $m_{\mathbb{Q}}(\chi) \leq 2$ and thus the only situation left to check is the one with $m_{\mathbb{Q}}(\chi) = 2$.

The following lemma characterizes the existence of a real minimal splitting field:

Lemma 3.3. *Let χ be an irreducible real valued character of G with $m_{\mathbb{Q}}(\chi) = 2$. Let $K = \mathbb{Q}(\chi) \subseteq \mathbb{R}$ and $\rho : G \to \mathrm{GL}_{2n}(K)$ a representation with character 2χ. Then the following statements are equivalent:*

(1) There exists a minimal splitting field $F \subseteq \mathbb{R}$.

(2) There exists a G-isomorphism $f : K^{2n} \to K^{2n}$ with $f^2 = \kappa \mathbb{1}_{K^{2n}}$, $\kappa > 0$ and $\sqrt{\kappa} \notin K$.

Proof. First we show that (1) implies (2). Let F be such a minimal splitting field, so $F = K(\sqrt{d})$ with $d \notin K^2$, $d > 0$ and there exists an F-representation ρ_0 which affords the character χ. Denote by σ the nontrivial element of $\mathrm{Gal}(F, K)$. By using the same techniques as in the proof of Theorem 3.1, we can show that there exists a matrix P such that

$$P^{-1}\rho P = \begin{pmatrix} \rho_0 & 0 \\ 0 & \sigma(\rho_0) \end{pmatrix},$$

with $\sigma(P) = P K_\sigma^{\otimes n}$. Now consider the matrix

$$C = \begin{pmatrix} \sqrt{d}\mathbb{1}_n & 0 \\ 0 & \sigma(\sqrt{d})\mathbb{1}_n \end{pmatrix},$$

then it follows, just like in Proposition 3.2, that PCP^{-1} has coefficients in K and commutes with the representation ρ. It's easy to check that the linear map f induced by this matrix satisfies all the wanted properties.

Next we prove that the existence of f gives us a minimal splitting field. Take $F = K(\sqrt{\kappa})$ and σ as before. Consider the representation ρ as an F-representation by extending the scalars, in the same way the map f is also an G-isomorphism over F. Since $f^2 = \kappa \mathbb{1}_{F^{2n}}$, we know that f can only have two different eigenvalues, namely $\sqrt{\kappa}$ and $-\sqrt{\kappa}$. Each of those eigenvalues occurs, since if v is an eigenvector for $\sqrt{\kappa}$, then $\sigma(v)$ is an eigenvector for $\sigma(\sqrt{\kappa}) = -\sqrt{\kappa}$ and vice versa. Now the splitting of F^{2n} into the eigenspaces of f gives us a decomposition of F^{2n} into two G-invariant subspaces and thus F is a splitting field. Because $F \subseteq \mathbb{R}$, this ends our proof. $\square$

We use this lemma to prove the existence of a real minimal splitting field:

Theorem 3.4. *Let $\chi \in \mathrm{Irr}(G)$ be an irreducible character which is afforded by a real representation. Then there exists a real minimal splitting field for χ.*

Proof. We use some ideas of the proof of [18, Theorem 31], but in our case, we work over a finite field extension of $\mathbb{Q}$ instead of over $\mathbb{C}$. Let $K = \mathbb{Q}(\chi)$, then we know that K is a real field extension of $\mathbb{Q}$. As mentioned above, we only have to check the case where $m_{\mathbb{Q}}(\chi) = 2$ because of the Brauer-Speiser Theorem. So there exists a field $F \supseteq K$ such that χ is afforded by a F-representation ρ and $[F : K] = 2$. If $F \subseteq \mathbb{R}$, there is nothing to prove, so we can assume that $F = K(\theta)$ with $\theta = \sqrt{d}$, $d \in K$ and $d < 0$.

Recall that ρ also induces a representation on the dual vector space V^*, namely $\rho^* : G \to GL(V^*)$ with

$$\rho_g^*(\varphi) = \varphi \circ (\rho_g)^{-1} = \varphi \circ \rho_{g^{-1}} \quad \forall \varphi \in V^*.$$

Since the values of χ lie in $\mathbb{R}$, the character of ρ^* is equal to χ. So the representations ρ and ρ^* are equivalent and there exists a G-isomorphism $f : V \to V^*$. Now look at the map

$$B : V \times V \to F : (v, w) \mapsto f(v)(w).$$

It is easy to see that B is a nondegenerate bilinear form on V. Since f is a G-isomorphism, B is clearly G-invariant, i.e. $B(\rho_g(v), \rho_g(w)) = B(v, w)$ for all $g \in G$, $v, w \in V$. This construction gives us an isomorphism between the space of G-morphisms from V to V^* and the space of G-invariant bilinear forms on V. Since V and V^* are irreducible, the space of G-morphisms between them has dimension 1. Hence, this G-invariant isomorphism f is unique up to scalar multiplication.

We can also consider the spaces $V_{\mathbb{C}} = \mathbb{C} \otimes_F V$ and $(V_{\mathbb{C}})^* = (V^*)_{\mathbb{C}} = \mathbb{C} \otimes_F V^*$ and the induced representations $\rho_{\mathbb{C}} : G \to \mathrm{GL}(V_{\mathbb{C}})$ and $\rho_{\mathbb{C}}^* : G \to \mathrm{GL}(V_{\mathbb{C}}^*)$. Then f extends to a G-isomorphism $f_{\mathbb{C}} : V_{\mathbb{C}} \to V_{\mathbb{C}}^*$ and the bilinear form B extends to $B_{\mathbb{C}} : V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C} : (v, w) \mapsto f_{\mathbb{C}}(v)(w)$. Since χ is afforded by a $\mathbb{R}$-representation, there exists also a nondegenerate symmetric bilinear form over $\mathbb{C}$ which is invariant under G by [18, Theorem 31]. Because of the uniqueness of nondegenerate G-invariant bilinear forms up to scalar multiplication, it now follows that $B_{\mathbb{C}}$, and hence also B, must be symmetric.

Now choose a G-invariant, positive definite, hermitian scalar product

$$\langle \, , \, \rangle : V \times V \to F.$$

For every $v \in V$, there exists a unique $\psi(v) \in V$ such that $B(w, v) = \langle w, \psi(v) \rangle$ for all $w \in V$. An easy computation then shows that ψ is bijective and antilinear, i.e. $\psi(\lambda v) = \overline{\lambda} \psi(v)$ for all $\lambda \in F$. Also, ψ is G-invariant because both B and $\langle \, , \, \rangle$ are G-invariant. So ψ^2 is a G-automorphism of V and thus $\psi^2 = \mu \mathbb{1}_V$ for some $\mu \in F$. By looking at V as a vector space over K, the map ψ becomes linear (since $K \subseteq \mathbb{R}$). Therefore, it is sufficient to show that $\mu \in K, \mu > 0$ and $\sqrt{\mu} \notin K$ because of Lemma 3.3.

For any $v, w \in V$ we have that

$$\langle v, \psi(w) \rangle = B(v, w) = B(w, v) = \langle w, \psi(v) \rangle = \overline{\langle \psi(v), w \rangle}.$$

Using this identity, we find for all $v, w \in V$ that

$$\overline{\mu} \langle v, w \rangle = \langle v, \mu w \rangle = \langle v, \psi^2(w) \rangle = \overline{\langle \psi(v), \psi(w) \rangle} = \langle \psi^2(v), w \rangle = \mu \langle v, w \rangle$$

and thus $\mu \in \mathbb{R} \cap F = K$. From the same computation with $v = w$, we also have that $\mu \langle v, v \rangle = \langle \psi(v), \psi(v) \rangle$ and $\mu > 0$ because $\langle \, , \, \rangle$ is positive definite. It's easy to check that $\sqrt{\mu} \notin K$ because χ cannot be afforded by a K-representation. This ends the proof. $\square$

3.3 Existence of c-hyperbolic units in number fields

In the previous parts we discussed the minimal splitting field for a representation. It is now important to know if we can find units in these field which are sufficiently hyperbolic.

Let $K = \mathbb{Q}(\theta)$ be any number field of degree n with monomorphisms $\sigma_i : K \to \mathbb{C}$ for $i \in \{1, \ldots, n\}$. Then we denote by $\mathcal{O}_K \subseteq K$ the subring of algebraic integers and U_K the group of units in $\mathcal{O}_K$. Note that the elements of U_K are exactly those $\mu \in K$ which have a minimal polynomial over $\mathbb{Q}$ with integer coefficients and unit constant term. We are interested in so-called c-hyperbolic elements of U_K:

Definition 3.5. *An algebraic unit $\mu \in U_K$ is called c-hyperbolic if*

$$\forall k \in \{1, \ldots, c\}, \ \forall i_1, i_2, \ldots, i_k \in \{1, \ldots, n\} : |\sigma_{i_1}(\mu) \ldots \sigma_{i_k}(\mu)| \neq 1.$$

The notion of a c-hyperbolic unit is the translation of c-hyperbolic integer-like matrices to number fields. By definition an element of K is c-hyperbolic if and only if the companion matrix of its minimal polynomial over $\mathbb{Q}$ is c-hyperbolic and integer-like. For any $x \in U_K$, we have that the product of all $\sigma_i(x)$ is up to sign equal to the constant term of the minimal polynomial of x over $\mathbb{Q}$ and therefore U_K cannot have n-hyperbolic elements. In the case where K is totally imaginary one can conclude in the same way that there are no units which are $\frac{n}{2}$-hyperbolic.

The structure of the group U_K is well known because of Dirichlet's Units Theorem. We will use the techniques of this theorem to investigate the existence of c-hyperbolic units in different number fields. Just as before, we distinguish between two possible cases, depending on wether or not our number field is totally imaginary.

Proposition 3.6. *Let K be a number field of degree n which is not totally imaginary. Then there exists a c-hyperbolic $\mu \in U_K$ for all $c \leq n - 1$.*

The proof is analogous to that in [6] in the case of a totally real Galois extensions over $\mathbb{Q}$.

Proof. It is of course sufficient to prove this for $c = n - 1$. Write $n = s + 2t$ as before with $s \neq 0$ by the conditions of the theorem. Assume that the first s monomorphisms $\sigma_1, \ldots, \sigma_s$ are the real ones. Look at the map

$$
\begin{aligned}
l : U_K &\rightarrow \mathbb{R}^{s+t} : \\
\mu &\mapsto (\log|\sigma_1(\mu)|, \ldots, \log|\sigma_s(\mu)|, \log|\sigma_{s+1}(\mu)|, \ldots, \log|\sigma_{s+t}(\mu)|),
\end{aligned}
$$

which is also used in the proof of Dirichlet's Units Theorem (see [19]). Now $l(U_K)$ is a lattice of dimension $s + t - 1$ in $\mathbb{R}^{s+t}$, which spans the vector space

$$V = \{(x_1, \ldots, x_{s+t}) \in \mathbb{R}^{s+t} \mid \sum_{i=1}^{s} x_i + \sum_{j=s+1}^{t} 2x_j = 0\}.$$

We claim that there exists an element $x \in l(U_K)$ for which

$$\forall k \in \{1, \ldots, c\}, \ \forall i_1, \ldots, i_k \in \{1, \ldots, s+t\} : \ \sum_{j=1}^{k} x_{i_j} \neq 0.$$

Any element $\mu \in U_K$ with $l(\mu) = x$ is then obviously a c-hyperbolic element.

It's easy to see that each of the equations $\sum_{j=1}^{k} x_{i_j} = 0$ is linearly independent of the equation determining V. For this to be true, we indeed use the assumption that $c \leq n - 1 = s + 2t - 1$ and $s > 0$. So the set of solutions of $\sum_{j=1}^{k} x_{i_j} = 0$ in V determines a subspace of V of dimension $s + t - 2$. It is not so difficult to see that a cocompact lattice of a vector space can never be contained in a finite union of proper subspaces. So this means that we can always find an x as described above. $\square$

Proposition 3.7. *Let K be a number field of degree n which is totally imaginary. Then there exists a c-hyperbolic $\mu \in U_K$ for all $c \leq \frac{n}{2} - 1$.*

The proof is completely the same as in the previous case and thus is left for the reader. As was stated above, the bounds given in the propositions are optimal.

Remark. *From the proof of Proposition 3.6 it follows that in a not totally imaginary number field K of degree n, we can always find a c-hyperbolic algebraic unit μ (for any $c < n$) such that $|\mu| > 1$, but all other conjugates have absolute value strictly smaller than 1. So we can assume that μ is a so called Pisot number.*

3.4 Existence of Galois extensions

In the previous part we investigated the existence of c-hyperbolic units in number fields. In the proof of the main theorem, it will be necessary to take fields extensions to ensure the existence of such units for the given c. The following theorem shows us that we can always find a Galois extension for any given degree:

Theorem 3.8. *Let $\mathbb{Q} \subseteq F$ be a finite degree field extension and $m \in \mathbb{N}_0$ a natural number. Then there exists a Galois extension $F \subseteq E$ such that $[E : F] = m$. If F is real, then we can find such an E which is not totally imaginary.*

Proof. Let n be the degree of the field extension $\mathbb{Q} \subseteq F$. From [6, Theorem 2.3] we know that there exists a totally real field extension $\mathbb{Q} \subseteq K$ for every given degree such that K is Galois over $\mathbb{Q}$ and $\mathrm{Gal}(K, \mathbb{Q})$ is abelian. In fact, the theorem doesn't say anything about the Galois group, but it follows from the proof that we can always assume this.

Let K be such a field extension with $[K : \mathbb{Q}] = mn$. Look at the field $E = KF$, the smallest field that contains both K and F. Since K is a splitting field over $\mathbb{Q}$ for a polynomial $f \in \mathbb{Q}[X]$, E is also a splitting field for the same polynomial f over the field F. Thus E is a Galois extension of F and $m \mid [E : F]$.

We now claim that $\mathrm{Gal}(E, F)$ is abelian. Look at the homomorphism

$$\pi : \mathrm{Gal}(E, F) \to \mathrm{Gal}(K, \mathbb{Q}) : \sigma \mapsto \sigma|_K,$$

then it is easy to see that π is injective. Since $\mathrm{Gal}(K, \mathbb{Q})$ is abelian, therefore also $\mathrm{Gal}(E, F)$ must be abelian.

Since $\mathrm{Gal}(E, F)$ is a finite abelian group and $m \mid |\mathrm{Gal}(E, F)|$, we can always find a (normal) subgroup $H \leq \mathrm{Gal}(E, F)$ of index m. By the fundamental theorem of Galois theory, there exists a subfield E_0 of E, Galois over F, such that $[E_0 : F] = m$ and thus the field E satisfies the conditions of the theorem. If F is real, then since K is also real, we have that E and so also E_0 is realized as a subfield of $\mathbb{R}$. $\square$

If F cannot be embedded in $\mathbb{R}$, then of course any field extension cannot be imbedded in $\mathbb{R}$. So the condition in this theorem is necessary.

3.5 Proving the last part of the main theorem

For constructing matrices that commute with a representation, the following trivial observation is useful:

Lemma 3.9. *Let $M, C_1, \ldots, C_m$ be $k \times k$ matrices over any field F and assume that C_i commutes with M for all i. Then the $km \times km$ matrices*

$$A = \begin{pmatrix} 0 & 0 & \ldots & 0 & C_1 \\ \mathbb{1}_k & 0 & \ldots & 0 & C_2 \\ 0 & \mathbb{1}_k & \ldots & 0 & C_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \mathbb{1}_k & C_m \end{pmatrix}, \ B = \begin{pmatrix} M & 0 & \ldots & 0 \\ 0 & M & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & M \end{pmatrix}$$

commute.

We are now ready to prove the main result of this paper.

Proof of Theorem 1.2. At the end of section 2, we already showed how to prove the first direction of this theorem.

Now let ρ be a $\mathbb{Q}$-irreducible representation that splits in more than $\frac{c}{m}$ components over $\mathbb{R}$. We construct a c-hyperbolic, integer-like matrix $C \in GL(\mathbb{Q})$ which commutes with $m\rho = \rho \oplus \ldots \oplus \rho$. Let χ be the character of an irreducible component of ρ. Take a field extension $\mathbb{Q} \subseteq F$ of minimal degree n such that χ is afforded by a F-representation $\rho_0 : G \to \mathrm{GL}_k(F)$. Note that n is also the number of components of ρ over $\mathbb{C}$. We first look for a c-hyperbolic unit μ in some field extension of F. Let E be a field extension like in Theorem 3.8 of degree m.

First assume that χ is not afforded by a real representation, so F, and hence also E, is totally imaginary. Since n is also the number of components of ρ over $\mathbb{C}$, we find that $n > 2\frac{c}{m}$ and thus $[E : \mathbb{Q}] > 2c \ (\Rightarrow [E : \mathbb{Q}] \geq 2c + 2)$. It follows from Proposition 3.7 that there exits a c-hyperbolic unit μ in E.

In the other case, χ is afforded by a real representation. Because of Theorem 3.4, we can assume that F is real and we can take E to be not totally imaginary. This time we have that $mn > c$ and thus there exists a c-hyperbolic unit μ in E as well. So in both cases we find a Galois extension E of F of degree m and a c-hyperbolic unit in E.

Let $\sigma_1, \ldots, \sigma_n$ be the n distinct monomorphisms of $F \to \mathbb{C}$ and take a matrix P as in Theorem 3.1, with corresponding representation $\tilde{\rho} = P^{-1}\rho P$ just as in the proof. Look at the matrix

$$R = \mathbb{1}_m \otimes P = \begin{pmatrix} P & 0 & \cdots & 0 \\ 0 & P & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P \end{pmatrix},$$

then it holds that

$$R(m\tilde{\rho})R^{-1} = \begin{pmatrix} P\tilde{\rho}P^{-1} & 0 & \cdots & 0 \\ 0 & P\tilde{\rho}P^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P\tilde{\rho}P^{-1} \end{pmatrix} = m\rho.$$

Let

$$f_0(X) = \prod_{\sigma \in \mathrm{Gal}(E,F)} (X - \sigma(\mu)) = \sum_{j=0}^{m} b_j X^j \in F[X].$$

Take the polynomials $f_i = \sigma_i(f) = \sum_{j=0}^{m} a_{ij} X^j$ and form the matrices

$$C_j = \begin{pmatrix} a_{1j} & 0 & \ldots & 0 \\ 0 & a_{2j} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{nj} \end{pmatrix} \otimes \mathbb{1}_k = \begin{pmatrix} \sigma_1(b_j) & 0 & \ldots & 0 \\ 0 & \sigma_2(b_j) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \sigma_n(b_j) \end{pmatrix} \otimes \mathbb{1}_k.$$

It's easy to see that every matrix C_j commutes with the representation $\tilde{\rho}$ and that $PC_jP^{-1} \in \mathrm{GL}_{kn}(\mathbb{Q})$ because of Proposition 3.2. Now construct the matrix

$$\tilde{C} = \begin{pmatrix} 0 & 0 & \ldots & 0 & -C_0 \\ \mathbb{1}_{kn} & 0 & \ldots & 0 & -C_1 \\ 0 & \mathbb{1}_{kn} & \ldots & 0 & -C_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \mathbb{1}_{kn} & -C_{m-1} \end{pmatrix},$$

then it is obvious that $\tilde{C}$ commutes with $m\tilde{\rho}$ because of Lemma 3.9. A direct computation shows that

$$
R\tilde{C}R^{-1} = \begin{pmatrix} 0 & 0 & \dots & 0 & -PC_0P^{-1} \\ PP^{-1} & 0 & \dots & 0 & -PC_1P^{-1} \\ 0 & PP^{-1} & \dots & 0 & -PC_2P^{-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & PP^{-1} & -PC_{m-1}P^{-1} \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 & 0 & \dots & 0 & -PC_0P^{-1} \\ \mathbb{1}_{kn} & 0 & \dots & 0 & -PC_1P^{-1} \\ 0 & \mathbb{1}_{kn} & \dots & 0 & -PC_2P^{-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \mathbb{1}_{kn} & -PC_{m-1}P^{-1} \end{pmatrix}
$$

and thus $R\tilde{C}R^{-1} \in \mathrm{GL}_{knm}(\mathbb{Q})$. Note that the characteristic polynomial $f(X)$ of $\tilde{C}$ equals $\left(\prod_{i=1}^{n} f_i(X)\right)^k$. It's easy to check that the polynomial f has coefficients in $\mathbb{Q}$ and all of its roots are conjugates of μ. This means that f is some power of the minimal polynomial of μ. Therefore $\tilde{C}$ is c-hyperbolic and integer-like because of our choice of μ. Since C satisfies all conditions of the theorem, this completes the other direction of the main theorem. $\square$

4 Generalization to other classes of infra–nilmanifolds

Up till now, we discussed infra–nilmanifolds modeled on a free c–step nilpotent Lie group, but in fact the results of this paper do generalize quite immediately to other classes of infra–nilmanifolds.

For example we can consider the Lie algebra $\mathfrak{g}_{c,d,r}$ which is the free c-step nilpotent and d–step solvable Lie algebra on r generators (over $\mathbb{R}$) and let $G_{c,d,r}$ be the corresponding simply connected Lie group. Theorem B. of [5] (which was slightly reformulated in [7, Theorem 2.1]) can now also be stated for manifolds modeled on $G_{c,d,r}$:

Theorem 4.1. *Let M be an infra-nilmanifold modeled on $G_{c,d,r}$, with holonomy group H and associated abelianized rational holonomy representation $\overline{\varphi} : H \to \mathrm{GL}_r(\mathbb{Q})$. Then the following are equivalent:*

M admits an Anosov diffeomorphism.

$\Updownarrow$

There exists an integer-like c-hyperbolic matrix $C \in \mathrm{GL}_r(\mathbb{Q})$ that commutes with every element of $\overline{\varphi}(H)$.

To prove this theorem one can follow almost word by word the original proof in [5].

Having obtained this theorem, we now also get the following generalization of our main result for free:

Theorem 4.2. *Let M be an infra-nilmanifold modeled on $G_{c,d,r}$, with holonomy group H and associated abelianized rational holonomy representation $\overline{\varphi} : H \to \mathrm{GL}_r(\mathbb{Q})$. Then the following are equivalent:*

M admits an Anosov diffeomorphism.

$\Updownarrow$

Every $\mathbb{Q}$-irreducible component of $\bar{\varphi}$ that occurs with multiplicity m, splits in more than $\frac{c}{m}$ components when seen as a representation over $\mathbb{R}$.

References

[1] Dani, S. G. and Mainkar, M. G. *Anosov automorphisms on compact nilmanifolds associated with graphs.* Trans. Amer. Math. Soc., 2005, 357 6, 2235–2251.

[2] Dekimpe, K. *Hyperbolic automorphisms and Anosov diffeomorphisms on nilmanifolds.* Trans. Amer. Math. Soc., 2001, 353 7, pp. 2859–2877.

[3] Dekimpe, K. *What an infra-nilmanifold endomorphism really should be . . . .* Topol. Methods Nonlinear Anal., 2012, 40 1, 111–136.

[4] Dekimpe, K. and Deschamps, S. *Anosov diffeomorphisms on a class of 2-step nilmanifolds.* Glasgow. Math. J., 2003, 45 pp. 2269–280.

[5] Dekimpe, K. and Verheyen, K. *Anosov diffeomorphisms on infra-nilmanifolds modeled on a free 2-step nilpotent Lie group.* Groups, Geometry and Dynamics, 2009, 3 4, pp. 555–578.

[6] Dekimpe, K. and Verheyen, K. *Anosov diffeomorphisms on nilmanifolds modeled on a free nilpotent Lie group.* Dynamical Systems – an international journal, 2009, 24 1, pp. 117–121.

[7] Dekimpe, K. and Verheyen, K. *Constructing infra-nilmanifolds admitting an Anosov diffeomorphism.* Adv. Math., 2011, 228 6, 3300–3319.

[8] Farrell, F. T. and Gogolev, A. *Anosov diffeomorphisms constructed from $\pi_k(\mathrm{Diff}(S^n))$.* J. Topol., 2012, 5 2, 276–292.

[9] Horn, R. A. and Johnson, C. R. *Matrix analysis.* Cambridge University Press, 1996.

[10] Isaacs, I. M. *Character theory of finite groups*, volume 69 of *Pure and Applied Mathematics*. New York-San Francisco-London: Academic Press, 1976.

[11] Lauret, J. *Examples of Anosov diffeomorphisms.* J. Algebra, 2003, 262 1, 201–209.

[12] Lauret, J. and Will, C. E. *Nilmanifolds of dimension ≤ 8 admitting Anosov diffeomorphisms.* Trans. Amer. Math. Soc., 2009, 361 5, 2377–2395.

[13] Mainkar, M. G. *Anosov automorphisms on certain classes of nilmanifolds.* Glasg. Math. J., 2006, 48 1, 161–170.

[14] Mainkar, M. G. and Will, C. E. *Examples of Anosov Lie algebras.* Discrete Contin. Dyn. Syst., 2007, 18 1, 39–52.

[15] Malfait, W. *Anosov diffeomorphisms on nilmanifolds of dimension at most six.* Geometriae Dedicata, 2000, 79 (3), 291–298.

[16] Payne, T. L. *Anosov automorphisms of nilpotent Lie algebras.* J. Mod. Dyn., 2009, 3 1, 121–158.

[17] Porteous, H. L. *Anosov diffeomorphisms of flat manifolds.* Topology, 1972, 11, pp. 307–315.

[18] Serre, J.-P. *Linear representations of finite groups*, volume 42 of *Graduate Texts in Math.* New York-Heidelberg-Berling: Springer–Verlag, 1977.

[19] Stewart, I. and Tall, D. *Algebraic Number Theory, second edition.* Chapman and Hall Mathematics Series. Chapman and Hall, London, 1987.